ON SUPERSIMPLE GROUPS

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Abstract. We show that an infinite group having a supersimple theory has a finite series of definable subgroups whose factors are infinite and either virtually FC or virtually simple modulo a finite FC-centre. We deduce that a group which is type-definable in a supersimple theory has a finite series of relatively definable subgroups whose factors are either abelian or simple groups. In this decomposition, the non-abelian simple factors are unique up to isomorphism.

1. Introduction

Model theory is the study of *definable sets*, that is, sets defined by a first order formula in a given language. It can be thought of as a generalisation of algebraic geometry where the objects under study are algebraic varieties: sets defined by systems of polynomial equations; in this case, the language considered is the language of fields which consists of the two function symbols $+$, $\times$ and the two constants 0 and 1. The existence of a notion of dimension on a class of definable sets strongly restricts the behavior of this class. In a linear algebraic group over an algebraically closed field, the Zariski dimension of an algebraic variety arises from the Zariski topology. A *supersimple group* is a group whose definable sets are equipped with a notion of dimension, the so-called *SU-rank*, arising from the logic topology and taking ordinal values. The SU-rank extends the Lascar U-rank of superstable groups and the Morley rank of groups of finite Morley rank. It is thus a far reaching generalisation of the Zariski dimension for linear algebraic groups. Examples of supersimple groups include:

- finite groups;
- linear algebraic groups over algebraically closed fields;
- groups of finite Morley rank;
- abelian groups which are divisible, or of bounded exponent;
- $\aleph_0$-stable groups;
- superstable groups;
- simple linear algebraic groups over pseudofinite fields;
- simple pseudofinite groups (*i.e.* abstractly simple groups having a pseudofinite theory).

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Thanks to Wagner for improving an earlier version of Theorem 5.3.

On the other hand, the infinite cyclic group **Z**, non-abelian free groups on n generators and more generally non-elementary hyperbolic groups are not supersimple (see Corollary 4.10).

In spite of the strong analogy between the U-rank and the SU-rank, the theory of supersimple groups is not nearly as developed as it is for its superstable analogues. For instance, Berline and Lascar [BL] have shown that every superstable group has a definable abelian subgroup of the same cardinality. Their original argument has been much simplified since and has shrunk to a few lines (see [Wag00, Remark 5.4.11]). Still, it is unknown whether a supersimple group G has an infinite abelian subgroup. If H is an abelian subgroup of G, then there exists a definable subgroup E of G which is finite-by-abelian and contains H (see [EJMR] or [Mil1]). Moreover, if one writes the SU-rank of G in the form $\omega^\alpha.n+\beta$ with $0 \leq \beta < \omega^\alpha$, then one can provide that the SU-rank of E be at least ω^α. So, asking whether G has an infinite abelian subgroup is equivalent with asking whether G has a definable finite-by-abelian subgroup of SU-rank at least ω^α. Finite-by-abelian groups are *FC groups*, *i.e.* groups whose every conjugacy class is finite. In [Bau], Baudisch has shown that a superstable group has a finite series whose factors are either abelian or simple groups. Our main result is:

Theorem 1.1. *Let G be an infinite supersimple group. Then there is a finite chain of definable subgroups $1 = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$ such that every quotient H_{i+1}/H_i is infinite and either virtually FC or virtually simple modulo a finite FC-centre.*

Consequently, Baudish's result extends to supersimple groups. His arguments however do not adapt to the supersimple context, mainly for two reasons: first, the proof of [Bau] is based on a transfinite induction on the U-rank of G, the induction basis being Berline and Lascar's result that a superstable group of U-rank 1 is virtually abelian; second, [Bau] makes a strong use of the connected component of a superstable group; in a supersimple group, there is also a notion of connected component, but it heavily depends on the specific parameter set over which it is defined. So we have to work without the use of Berline and Lascar's result and without connected components.

For the development of a suitable version of the result by Berline and Lascar, our basic idea is provided by [Wag00, Remark 5.4.11]: a supersimple group of U-rank 1 either has an infinite abelian subgroup or is virtually simple modulo a finite FC-centre; this makes a proof by transfinite induction still possible. To avoid the use of connected components, we study just-infinite supersimple groups. These turn out to be either virtually FC or virtually simple.

Other important tools are Lascar's additive properties of the SU-rank, Wagner's version of Zilber's indecomposability Theorem for supersimple groups and the observation that the FC-centre of a supersimple group is defined by a first order formula. It should be mentioned that these methods mainly come from Berline and Lascar's investigation of superstable groups in [BL], later extended by Wagner to supersimple groups [Wag00].

The main theorem has several consequences. We outline three of them here. Sela has recently shown that the first order theory of a torsion-free hyperbolic group is stable, hence simple. Using a theorem of Delzant [Del], we derive:

Corollary 1.2. *A (non virtually cyclic) hyperbolic group is not supersimple.*

Jaligot, Elwes, Macpherson and Ryten have proved in [EJMR] that the soluble radical of a supersimple group of finite SU-rank satisfying an additional technical assumption is soluble and definable. This has been generalised to an arbitrary supersimple group in [Mil3]. A group G is said to be *type-definable* if it is defined by the conjunction of infinitely many first order formulae. If G is a type-definable group, a subset of G is called *relatively definable in* G if it is the intersection of G with a definable set. We show the following:

Corollary 1.3. *The soluble radical of a type-definable supersimple group G is solvable and relatively definable in G.*

Pseudofinite supersimple groups of finite SU-rank have been investigated in [EJMR]. Pseudofinite fields are supersimple of SU-rank one and pseudofinite simple groups are supersimple of finite SU-rank according to [Wil]. Using the work of Wilson [Wil] and Ryten [Ryt] on pseudofinite simple groups, we conclude that:

Corollary 1.4. *A pseudofinite supersimple group either interprets a pseudofinite field or is virtually solvable.*

2. Preliminaries

Let G be a group and let x be an element of G. We write x^G for the G*-conjugacy class* $\{g^{-1}xg : g \in G\}$ of x and $C(x)$ for its *centraliser* $\{g \in G : g^{-1}xg = x\}$ in G. If y is another element of G, we write $[x, y]$ for the commutator $x^{-1}y^{-1}xy$. The element x is said to be FC *in* G if x^G is finite. We call the subgroup consisting of the FC elements of G the FC*-centre* of G and we write it $FC(G)$. We say that G is an FC *group* if each of its elements is FC in G.

Two subgroups of G are called *commensurable* if their intersection has a finite index in each of them. The following theorem originally comes from [Sch], with some details taken from [Wag00, Theorem 4.2.4] and its proof.

Fact 2.1 (Schlichting [Sch])**.** *Let G be a group and H a subgroup of G such that $H/H \cap H^g$ remains finite and bounded by a natural number for all g in G. Then, there exists a normal subgroup N of G such that $H/H \cap N$ and $N/N \cap H$ are finite. There are four elements $g_1, \dots, g_4$ in G such that N is a subset of $H^{g_1}H^{g_2}H^{g_3}H^{g_4}$. Moreover, N is a finite extension of a finite intersection of G-conjugates of H. In particular, if H is definable, then so is N.*

We recall the properties of a supersimple group that will be needed in the sequel. A *supersimple group* G is a group equipped with a rank function defined on the set of all definable subsets of G and taking ordinal values. As this is the only notion of rank that we shall use in the paper, we simply write $rk(X)$ for the rank of such a set X. We refer to [Wag00, Definition 5.1.1] for the precise definition of this rank, which will not be needed here, and more generally to [Wag00, Chapter 5] for precisions on supersimple theories. We shall try to avoid technicalities and use only the basic properties of this rank that we recall now.

(1) *Monotonicity.* The rank is increasing: if $X \subset Y$ are two definable subsets of G, then $rk(X)$ is smaller than or equal to $rk(Y)$.
(2) *Definable invariance.* The rank is preserved by definable isomorphisms: if there is a definable isomorphism between two definable sets X and Y, then $rk(X)$ equals $rk(Y)$.

(3) *Interpretation and additivity.* If the group G is supersimple, then so is G^{eq}. We shall not need the precise definition of G^{eq}. Let us just say that G^{eq} is a multi-sorted structure built up out of G which allows us to treat factor groups by definable subgroups as definable sets. The only useful consequence for the paper is that any left or right quotient G/H by a definable subgroup H has an ordinal rank. The rank $rk(G/H)$ can be controlled in terms of $rk(H)$ and $rk(G)$ by Lascar inequalities (see Fact 2.2).
(4) *Rank zero sets.* A definable set (in G^{eq}) has rank zero if and only if it is finite. In particular, if H is a definable subgroup of G, then $rk(G/H)$ equals zero if and only if H has a finite index in G.
(5) *Preservation by elementary extension.* If the group G is supersimple, then so is each of its elementary extension. We recall that $\mathbf{G}$ is called an *elementary extension of* G if it contains G and for every formula $\varphi(x_1, \ldots, x_n)$ in the free variables $x_1, \ldots, x_n$ without parameters and all elements $g_1, \ldots, g_n$ of G, the formula $\varphi(g_1, \ldots, g_n)$ holds in G if and only if it holds in $\mathbf{G}$

We shall use a modicum of ordinal arithmetic: any ordinal α decomposes in base ω, which means that there are unique ordinals $\alpha_1 > \cdots > \alpha_n$ and non-zero natural numbers $k_1, \ldots, k_n$ such that α equals $\omega^{\alpha_1}.k_1 + \cdots + \omega^{\alpha_n}.k_n$. If α and β are two ordinals, we may assume that α equals $\omega^{\alpha_1}.k_1 + \cdots + \omega^{\alpha_n}.k_n$ and β equals $\omega^{\alpha_1}.\ell_1 + \cdots + \omega^{\alpha_n}.\ell_n$ for the same $\alpha_1, \ldots, \alpha_n$, adding some additional possibly zero k_i and ℓ_i if necessary. We call their *Cantor sum* and write $\alpha \oplus \beta$ for the ordinal defined by

$$\alpha \oplus \beta = \omega^{\alpha_1}.(k_1 + \ell_1) + \cdots + \omega^{\alpha_n}.(k_n + \ell_n)$$

We say that an ordinal is a *monomial* if it is of the form $\omega^{\alpha}.n$ where α is an ordinal and n a natural number. Note that for two monomials $\omega^{\alpha}.n$ and $\omega^{\beta}.m$, the two operations coincide:

$$\omega^{\alpha}.n + \omega^{\beta}.m = \omega^{\alpha}.n \oplus \omega^{\beta}.m$$

The following inequalities are due to Lascar and their analogue in the supersimple context can be found in [Wag00, Theorem 5.1.6].

Fact 2.2 (Lascar inequalities)**.** *Let G be a supersimple group and H a definable subgroup of G. Then*

$$rk(H) + rk(G/H) \leq rk(G) \leq rk(H) \oplus rk(G/H)$$

The following fact will be much used in the sequel:

Fact 2.3 ([Mil2, Neu])**.** *The FC-centre of a supersimple group G is definable and its derived subgroup G' is finite.*

We call a group G *FC-soluble* if it has a finite series $1 \lhd \cdots \lhd G$ whose factors are FC groups. Soluble-by-finite groups are FC-soluble groups. For supersimple groups, the converse is also true:

Fact 2.4 (Milliet [Mil2])**.** *In a supersimple group, any FC-soluble subgroup has a soluble subgroup of finite index. Any definable FC-soluble subgroup has a definable soluble subgroup of finite index.*

The following Fact is the definable version of [Wag01, Corollary 4.2] which is stated for hyperdefinable (hence type-definable) groups. As we could not find any precise reference, we give a proof here for the sake of completeness.

Fact 2.5 (Wagner [Wag01, Corollary 4.2])**.** *Let G be a supersimple group of rank $\omega^{\alpha_1}.k_1+\cdots+\omega^{\alpha_n}.k_n$ with $\alpha_1 > \cdots > \alpha_n$. Then for every natural number i such that $1 \leq i \leq n$, there is a definable normal subgroup H of G of rank $\omega^{\alpha_1}.k_1+\cdots+\omega^{\alpha_i}.k_i$. Moreover, if N is another subgroup of G satisfying the same properties as H, then N and H are commensurable.*

Proof. Without loss of generality, we may assume that G is κ-saturated for some cardinal κ and we write β_i for $\omega^{\alpha_1}.k_1 + \cdots + \omega^{\alpha_i}.k_i$. By [Wag01, Corollary 4.2], there is a type-definable normal subgroup H of G whose rank equals β_i. We recall that β_i is by definition the rank of each of the generic types of H. By [Wag01, Theorem 4.4], the group H is the conjunction of definable groups H_j for j running in some set J. We may close this family by finite intersections, then remove the members whose rank is not minimal, assume that the rank of every H_j equals β say and that the groups H_j are pairwise commensurable. It follows that for every j, the index of H in H_j is bounded by κ, so H is a generic type of H_j by [Wag00, Lemma 4.1.15]. Thus β equals β_i. Let us take any H_j. As H is normal in G, the groups H_j^g and H_j are commensurable for every g in G. By the Compactness Theorem and the saturation hypothesis, the cardinal of $H_j/H_j \cap H_j^g$ remains bounded by some natural number when g ranges over G. By Theorem 2.1, there is a definable normal subgroup N of G which is commensurable with H. Therefore, $rk(N)$ equals β_i.

If K is another definable group satisfying the desired requirements, then $K/K \cap N$ and $N/K \cap N$ are bounded by κ according to [Wag01, Corollary 4.2]. By the Compactness Theorem, K and N are commensurable. □

3. A Jordan-Hölder decomposition

We prove the main theorem in this section.

Lemma 3.1. *Let G be an infinite supersimple group. Then G has a series $1 = H_0 \lhd \cdots \lhd H_n = G$ of definable subgroups such that*

(1) *the rank of every factor H_{i+1}/H_i is a non-zero monomial, say $\omega^{\alpha_i}.k_i$,*
(2) *for every i in $\{0, \ldots, n-1\}$, the rank of any definable normal subgroup of H_{i+1}/H_i is either equal to $\omega^{\alpha_i}.k_i$ or strictly less than ω^{α_i}.*

Proof. We proceed by induction on the rank of G.

If the rank of G is 1, we take the series $1 \lhd G$.

Let us assume that the rank of G equals α and that the lemma is proved for every group whose rank is less than α.

If α is a monomial, we write it say as $\omega^\beta.n$. Then, either the chain $1 \lhd G$ satisfies the requirements of Lemma 3.1, or there is a definable normal subgroup H whose rank is of the form $\omega^\beta.k+\gamma$ with $1 \leq k < n$ and $\gamma < \omega^\beta$. By Lascar inequalities, the ranks of both H and G/H are strictly less than α. We may apply the induction hypothesis to H and G/H and find two series $1 = H_0 \lhd \cdots \lhd H_n = H$ and $1 = H_n/H \lhd \cdots \lhd H_{n+r}/H = G/H$ with the required properties. As groups, $H_{n+i+1}/H/H_{n+i}/H$ and H_{n+i+1}/H_{n+i} are definably isomorphic, so they must have the same rank; moreover, their definable normal subgroups are in a (rank preserving) one-to-one definable correspondence, so that the series $1 = H_0 \lhd \cdots \lhd H_{n+r} = G$ meets our requirements.

If α is not a monomial, it can be written in the form $\omega^\beta.n+\gamma$ with $0<\gamma<\omega^\beta$. By Proposition 2.5, the group G has a definable normal subgroup H of rank $\omega^\beta.n$. By Lascar inequalities, both the rank of H and the rank of G/H are strictly less than α, so we conclude again by applying the induction hypothesis as in the monomial case. □

To prove Fact 3.2, we shall need Zilber's Indecomposability Theorem formulated by Wagner for the groups which are hyperdefinable in a supersimple theory (see [Wag00, Theorem 5.4.5]). We give the appropriate version for definable groups here. Again, we could not find a precise reference for this particular version, so we provide a proof bellow.

Fact 3.2 (Wagner's version of Zilber's Indecomposability Theorem)**.** *Let G be a supersimple group and let $\mathfrak{X}$ be a collection of definable subsets of G. If the rank of G is strictly less than $\omega^{\alpha+1}$, then there is a definable subgroup H of G included in $X_1^{\pm1}\cdots X_n^{\pm1}$ for some $X_1,\ldots,X_n$ belonging to $\mathfrak{X}$ such that $rk(XH)<rk(H)+\omega^\alpha$ holds for all X in $\mathfrak{X}$. Moreover, H^g equals H for every element g of G stabilising $\mathfrak{X}$ setwise by conjugation.*

Proof. Without loss of generality, we assume that G is κ-saturated for some cardinal κ. By [Wag00, Theorem 5.4.5], there is a type-definable subgroup T of G included in $X_1^{\pm1}\cdots X_n^{\pm1}$ for some $X_1,\ldots,X_n$ in $\mathfrak{X}$ such that $rk(XT)<rk(T)+\omega^\alpha$ holds for all X in $\mathfrak{X}$ and such that T is normalised by every element g of G stabilising $\mathfrak{X}$ setwise. By [Wag00, Corollary 5.5.4], the group T is a subgroup of a definable subgroup H of G having the same rank as T. By the Compactness Theorem, we may assume that H is still a subset of $X_1^{\pm1}\cdots X_n^{\pm1}$. Let A be the subset of G whose elements stabilise $\mathfrak{X}$ setwise by conjugation. As T is a subgroup of H and because T is invariant under conjugation by elements of A, the set of A-conjugates of H consists of commensurable uniformly definable subgroups: they must be uniformly commensurable. By Theorem 2.1, there is a definable subgroup N of G which is commensurable with H, is invariant under conjugation by elements of A and is a subset of a finite product of elements of $\mathfrak{X}\cup\mathfrak{X}^{-1}$. As $N/T\cap N$ and $T/T\cap N$ are bounded by κ, the groups T and N have the same rank by [Wag00, Lemma 4.1.15]. It follows that $rk(XT)$ equals $rk(XN)$ for any X in $\mathfrak{X}$, so that the inequality $rk(XN)<rk(N)+\omega^\alpha$ holds. □

Let G be a group and let P be a group property. We say that G is *virtually P* if G has a definable normal subgroup of finite index with property P. We say that G is *finite-by-P* if it has a finite normal subgroup N such that G/N has property P. We say that G is *finite-by-P-by-finite* if G has a series $1\lhd N\lhd H\lhd G$ such that N and G/H are finite and H/N has property P.

Lemma 3.3. *Let G be a supersimple group which is not virtually FC. If every non-trivial definable normal subgroup of G has a finite index, then every normal subgroup of G is definable (hence either is trivial or has a finite index).*

Proof. G must have a monomial rank say $\omega^\alpha.n$ for otherwise Proposition 2.5 together with Lascar inequalities would yield a normal definable subgroup of infinite index. If there is a conjugacy class g^G whose rank is strictly less than ω^α, then so is the rank of $G/C(g)$. By Lascar inequalities, the rank of $G/C(g)$ is zero so $C(g)$ has a finite index in G. If $g \neq 1$, then $FC(G)$ contains g hence is a non-trivial normal subgroup. $FC(G)$ is definable by Lemma 2.3 so it has a finite index in G. It follows that G is virtually FC, a contradiction. So, the rank of any non-trivial conjugacy class is at least ω^α. If N is a normal proper subgroup of G containing some $n \neq 1$, by Theorem 3.2 there is definable normal subgroup H such that $H \leq \langle n^G \rangle \leq N$ and $rk(n^G H) < rk(H) + \omega^\alpha$ hold. As $rk(n^G)$ is at least ω^α, the group H is non-trivial. It follows that H has a finite index in G, so N is a finite union of cosets of H: it is definable. □

Definition 3.4 (McCarthy, Magnus)**.** A group is called *just-infinite* if it is infinite and if each of its normal subgroups either is trivial or has a finite index.

Lemma 3.5. *Let G be a supersimple group. If G is just-infinite, then it is either virtually FC or virtually simple.*

Proof. Let us suppose that G is not virtually FC. Let G^0 be the intersection of every definable subgroup of G having a finite index. G^0 is a normal subgroup of G hence either is trivial or has a finite index. If G^0 has a finite index, it is simple by Lemma 3.3. Suppose for a contradiction that G^0 be trivial. We may use [Wag11]: G is residually finite hence virtually nilpotent, so it has a minimal nilpotent subgroup N of finite index. N is a characteristic subgroup. As a proper normal subgroup of G, the derived group N' is trivial, so G is virtually abelian, a contradiction. We may alternatively use a compactness argument: by Lemma 3.3, being just-infinite and being virtually simple are properties of the theory of G, so we may assume that G is $\aleph_0$-saturated. G^0 is the intersection of infinitely many definable normal subgroups of G having a finite index. As it is trivial, one can find a countable subchain of strictly decreasing definable subgroups $G_1, G_2, \ldots$ whose indexes in G are finite. Let us call A a countable parameter set over which every G_i is definable. Let G_A^0 be the intersection of all the subgroups of G of finite index which are definable using parameters in A. As G_A^0 is normal in G, either it has a finite index or it is trivial. Being a subgroup of $\bigcap_{i \in \mathbf{N}} G_i$, it must be trivial. This contradicts the Compactness Theorem and the $\aleph_0$-saturation of G. □

Lemma 3.6. *Let G be a supersimple group having a monomial rank $\omega^\alpha.n$. Assume that the rank of every definable normal subgroup of G is either equal to $\omega^\alpha.n$ or strictly less than ω^α. Then, the quotient $G/FC(G)$ is either finite or just-infinite.*

Proof. Let H be a definable normal subgroup of G whose rank is less than ω^α and let h be an element of H. Then H contains h^G so $rk(h^G) < \omega^\alpha$, hence $rk(G/C(h)) < \omega^\alpha$. By Lascar inequalities, $G/C(h)$ is finite. It follows that any normal subgroup whose rank is less than ω^α is a subgroup of $FC(G)$. If the rank of $FC(G)$ equals $\omega^\alpha.n$, then $G/FC(G)$ is finite. So, we may assume that the rank of $FC(G)$ is less than ω^α, so that $G/FC(G)$ has rank $\omega^\alpha.n$. Note that if $H/FC(G)$ is a definable normal subgroup of $G/FC(G)$, then H is a definable normal subgroup of G. Thus, every definable normal subgroup of $G/FC(G)$ must either be trivial or have maximal rank hence have a finite index. In this case, the group $G/FC(G)$ is just-infinite by Lemma 3.3. □

Theorem 3.7. *Let G be an infinite supersimple group. Then, there is a finite chain of definable subgroups $1 = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$ such that every quotient H_{i+1}/H_i is infinite and either virtually FC or virtually simple modulo a finite FC-centre.*

Proof. Lemma 3.1 provides a series $1 = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$. Lemma 3.6 and Lemma 3.5 ensure that there is a refinement of the series whose factors have the required properties. □

4. Corollaries of Theorem 3.7

A supersimple FC group is finite-by-abelian according to Lemma 2.3. As a finite group F decomposes into a finite series $1 = F_0 \lhd F_1 \lhd \cdots \lhd F_n = F$ whose factors are either abelian or simple, we may state:

Corollary 4.1. *Let G be a supersimple group. Then there is a finite series of definable subgroups $1 = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$ such that every factor H_{i+1}/H_i is either abelian or simple. Moreover, in this decomposition, the non-abelian simple factors (in particular the infinite simple ones) are uniquely determined up to a definable group isomorphism.*

Proof. We need just to prove that the non-abelian simple factors are unique. Let be two series for G whose factors are either abelian or simple. By the Schreier Refinement Theorem, in both of those two normal series of G, some intermediate subgroups can be inserted to yield two refined normal series for G whose factor groups coincide, counting their multiplicity, up to a group isomorphism. As neither an abelian group nor a simple one has a non-abelian simple proper factor, the non-abelian simple factors occurring in the two refined normal series must already occur in the original series. □

Theorem 3.7 has many consequences concerning soluble subnormal subgroups, residual properties and pseudofinite groups. We discuss them here.

4.1. About soluble subgroups and local properties.

Corollary 4.2. *Let G be a supersimple group. Then, either G interprets an infinite simple group or G is virtually soluble.*

Proof. If G does not interpret an infinite simple group, then G has a series whose factors are finite or abelian. So G has a series with FC factors: it is FC-soluble. By Theorem 2.4, G is virtually soluble. □

Let G be any group with a series $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$. If N is a normal subgroup of G, it has a induced series

$$1 = G_0 \cap N \lhd G_1 \cap N \lhd \cdots \lhd G_n \cap N = N$$

where every factor $G_{i+1} \cap N/G_i \cap N$ is isomorphic to $(G_{i+1} \cap N)G_i/G_i$ hence can be considered as a normal subgroup of G_{i+1}/G_i. We say that H is a *subnormal subgroup of G* if there is a finite series $H \lhd \cdots \lhd G$. For a subnormal subgroup H of G, an immediate induction yields an induced series (of length no greater than n) whose factors are isomorphic to subnormal factors of the original series for G. In particular:

Corollary 4.3. *For any supersimple group G, there exists a natural number n such that any subnormal subgroup S (not necessarily definable) has a series $1 = H_0 \lhd H_1 \lhd \cdots \lhd H_m = S$ whose length m is bounded by n and whose factors are either abelian or simple.*

Corollary 4.4. *For any supersimple group, there is a natural number n bounding the derived length of any subnormal soluble subgroup.*

Question 1. In a supersimple group, or even a superstable one, is there a bound on the derived lengths of soluble subgroups? Is the number of infinite abelian factors bounded? It would suffice to prove this in the case where the ambient group is infinite simple and has a monomial rank.

Corollary 4.5. *A locally soluble subnormal subgroup of a supersimple group is soluble.*

Proof. By [Rob], every locally soluble simple group is finite hence soluble. The use of Corollary 4.3 concludes the proof. □

By definition, the *soluble radical* of a group is the subgroup generated by all its normal soluble subgroups. It is a locally soluble characteristic subgroup. By Corollary 4.5, the soluble radical of a supersimple group is soluble. By an observation of Ould Houcine [Oul], the soluble radical of any group is definable provided that it is a soluble subgroup. This shows:

Corollary 4.6. *The soluble radical of a supersimple group G is a soluble and definable subgroup.*

Wagner and Evans [EW] have shown that a supersimple $\aleph_0$-categorical group is finite-by-abelian-by-finite. A simple $\aleph_0$-categorical group is known to be nilpotent-by-finite [Wag00]. Note that $\aleph_0$-categorical groups are locally finite groups of bounded exponent. In the same vein:

Corollary 4.7. *A locally finite supersimple group of bounded exponent is soluble-by-finite.*

Proof. The same argument as the one by Felgner in [Fel] works: by an inspection of the classification of finite simple groups and a result of Kargapolov [Kar]. □

Question 2. Is a locally finite supersimple group soluble-by-finite? This question is equivalent to asking whether a supersimple locally finite simple group is finite. Recall from [HK] that in an infinite locally finite simple group, the centraliser of every element is infinite.

4.2. **About free groups, hyperbolic groups and residual properties.** Non-abelian free groups are not superstable (a result of Gibone, see [Poi]). Sela has shown that free groups and more generally torsion-free hyperbolic groups are stable [Sel]. By a Theorem of Magnus, an early result of combinatorial group theory, free groups are residually-nilpotent (see [Hal, Chapter 11]). Wagner [Wag11] has shown that a supersimple group which is residually soluble is in fact soluble. It follows that non-abelian free groups are not supersimple. This can also be derived directly from Corollary 4.1 following Poizat's argument in [Bau].

Corollary 4.8. *A non-abelian free group is not supersimple.*

Proof. Suppose for a contradiction that F is a supersimple non-abelian free group on $r > 1$ generators. By Corollary 4.1, F has a series of bounded length n with abelian or simple factors. So does every quotient of F by a normal subgroup. In particular, any soluble group generated by r elements is isomorphic to a quotient of F and would have derived length bounded by n. $\square$

Let G be a (non virtually cyclic) hyperbolic group. A theorem of Gromov states that if g and h are non-commuting elements of G, then g^n and h^m generate a non-abelian free subgroup on 2 generators for sufficiently large natural numbers n and m (see [GH, Théorème 37 p. 157]). This was generalised by Delzant [Del]: if g is an element of G, we write $\ell(g)$ for the length of g and $n(g)$ for the limit of $\ell(g^n)/n$ when n goes to infinity. This limit can be shown to be a natural number.

Fact 4.9 (Delzant [Del])**.** *Let G be a (non virtually cyclic) δ-hyperbolic group. Then, there is a natural number m such that for all $g_1, \dots, g_n$ in G with $n(g_i) = n(g_j) \geq 1000\delta$, the normal subgroup generated by $g_1^n, \dots, g_m^n$ is free.*

Corollary 4.10. *A (non virtually cyclic) hyperbolic group is not supersimple.*

Let $\mathscr{C}$ be a *pseudo-variety* of groups *i.e.* a class of groups closed under taking subgroups, quotients and finite Cartesian products. A group G is said to be *residually* $\mathscr{C}$ if

$$\bigcap_{\substack{H \lhd G \\ G/H \in \mathscr{C}}} H = \{1\}$$

It was shown in [Wag11] that a supersimple group which is residually $\mathscr{C}$ has series $G = G_0 \rhd \cdots \rhd G_n$ such that every factor G_i/G_{i+1} belongs to $\mathscr{C}$ and G_n is nilpotent. In the same spirit:

Corollary 4.11. *Let $\mathscr{C}$ be a pseudo-variety of groups that does not contain an infinite simple group. A residually $\mathscr{C}$ supersimple group is virtually soluble. In particular, a residually soluble supersimple group is soluble (Wagner).*

Proof. Let G be residually $\mathscr{C}$ and supersimple. By Corollary 4.6, the soluble radical R of G is soluble and definable. It may not be equationnally definable, so we may not apply [Oul07, Lemma 2.2.2], but it is *equationnally type-definable*, that is to say, definable by an infinite set of equations, namely $f(x, g_1, \dots, g_{2^n}) = 1$ where the g_i range over all elements of G and where $f(x, g_1, \dots, g_{2^n})$ is the word defined inductively on $n \geq 1$ by putting

$$f(x, g_1, g_2) = [x^{g_1}, x^{g_2}] \quad \text{and}$$

$$f(x, g_1, g_2, \dots, g_{2^{n+1}}) = [f(x, g_1, g_2, \dots, g_{2^n}), f(x, g_{2^n+1}, g_{2^n+2}, \dots, g_{2^{n+1}})]$$

By [Wag11, Lemma 1], the quotient G/R is residually $\mathscr{C}$. If G/R is infinite, then, by Corollary 4.1, it has a finite series $1 = G_0/R \lhd \cdots \lhd G_n/R = G/R$ such that G_1/R is infinite and either finite-by-simple, or finite-by-abelian. The first case is impossible as $\mathscr{C}$ does not contain any infinite simple group. In the second case, there is a finite F extension of R such that G_1/F is abelian, so G_1/F is a subnormal abelian subgroup of G/F and is a subgroup of its *Baer radical*, that is, the subgroup generated by each of its subnormal abelian subgroups. It is a characteristic locally nilpotent group. By Corollary 4.5, G_1/F is a subgroup of a normal soluble subgroup S/F of G/F. As F is soluble-by-finite, S is FC-soluble hence soluble-by-finite by Theorem 2.4. It must have a maximal soluble subgroup of finite index which is characteristic in S hence normal in G. But S/R is infinite: this contradicts the maximality of R. It follows that G/R is finite. $\square$

4.3. **About pseudofinite groups.** Wilson has shown in [Wil] that every pseudofinite simple group is elementary equivalent to a Chevalley group over a pseudofinite field. As pseudofinite fields are supersimple (see [Hru]) and have rank 1 (see [CVM]), a pseudofinite simple group is supersimple and has a finite rank.

Corollary 4.12. *A pseudofinite supersimple group has a finite series whose factors are abelian, finite simple groups, or elementary equivalent to Chevalley groups over pseudofinite fields. In particular, if a pseudofinite supersimple group does not interpret a pseudofinite field, it must be virtually soluble.*

Proof. If a pseudofinite supersimple group G interprets an infinite simple group S, this must be a Chevalley group over a pseudofinite field F. By [Ryt], F is interpretable in G. Otherwise, G is virtually soluble by Corollary 4.2. $\square$

For a group G, let us say that G^{eq} *eliminates* $\exists^\infty$ if for any uniformly definable family of sets (in the sense of G^{eq}), there is an upper bound on the size of its finite members. Elwes, Jaligot, Macpherson and Ryten have shown in [EJMR] that a supersimple pseudofinite group G having rank 2 is virtually soluble provided that G^{eq} eliminates $\exists^\infty$. With this assumption, they also showed using [Wil] that there were no supersimple pseudofinite simple groups of rank less than 3 and that the ones having rank 3 are elementary equivalent to $PSL_2(F)$ for some pseudofinite field F.

Corollary 4.13. *A pseudofinite supersimple group G of rank 3 such that G^{eq} eliminates $\exists^\infty$ is either virtually soluble or has a series of definable subgroups $1 \lhd F \lhd S \lhd G$ such that F and G/S are finite and S/F is elementary equivalent to $PSL_2(F)$ for some pseudofinite field F.*

5. Type-definable version

Let T be be first order theory equipped with a definable binary law $\times$. Let n be a natural number. We call a *type-definable group* a partial type π in n variables such that for all model M of T, the set of realisations of π in the Cartesian product M^n has a group structure for the law $\times$. If the structure M is saturated, we shall identify the type π and the group that it defines on M^n. We now extend the main results of the previous section to type-definable groups in a supersimple theory. We first recall:

Fact 5.1 (Wagner [Wag00, Theorem 5.5.4])**.** *In a supersimple theory, a type-definable group is a conjunction of definable groups.*

It is shown in [Bau, Corollary 1.11] that a type-definable simple group in a superstable theory is definable. We generalise that to supersimple theories:

Proposition 5.2. *A simple group H which is type-definable in a supersimple theory is definable.*

Proof. We may assume that H is infinite. We may also add some parameters to the language and assume that H is defined with no parameters. By [Wag00, Proposition 5.4.9], any elementary extension of H is still a simple group, so we may assume that the ambient structure is $\aleph_0$-saturated. By Fact 5.1, the type-definable group H is contained in a group G which is definable without parameters and that we may take of the minimal possible rank, so that the cardinal of G/H is bounded by κ. Let us consider the intersection G^0 of every subgroup of G having finite index and being definable without parameters. G^0 is a non-trivial subgroup of H which is normalised by G. Because H is simple, G^0 equals H. It follows that H is normal in G. Note that both G and H have a monomial rank say equal to $\omega^\alpha.n$. Let $h \neq 1$ be an element of H. If the rank of h^G is strictly less than ω^α, then $C(h)$ has a finite index in G so h belongs to $FC(H)$, a contradiction with H being infinite and simple. So the rank of h^G is at least ω^α. By Fact 3.2, there is a definable normal subgroup N of G such that $N \leq \langle h^G \rangle \leq H$ and $rk(h^G N) < rk(N) + \omega^\alpha$ hold. As we have $rk(h^G) \geq \omega^\alpha$, the group N is non-trivial and must equal H by simplicity of H. □

Let G be a type-definable group and let H be a type-definable subgroup of G. We say that H is *relatively definable in G* if H is the intersection of G with a definable set. We now extend Theorem 3.7 to type-definable groups:

Theorem 5.3. *Let G be a type-definable supersimple group. There is a finite chain of relatively definable subgroups $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n \lhd G$ such that every quotient G_{i+1}/G_i is either abelian or simple. Moreover, the non-abelian simple factors are unique up to isomorphism.*

Proof. We may add parameters to the language, assume that G is defined with no parameters and that the structure is $\aleph_0$-saturated. By Fact 5.1, the group G is a subgroup of some group H definable with no parameters. We may assume that H has a minimal rank so that H/G is bounded. Let H^0 be the intersection of every subgroups of finite index which are definable with no parameters. H^0 is a normal subgroup of H included in G. Let $1 = H_0 \lhd H_1 \lhd \cdots \lhd H_n = H$ be a definable series for H provided by Theorem 3.7. For every i, as $H_{i+1} \cap H^0/H_i \cap H^0$ is isomorphic to $(H^0 \cap H_{i+1})H_i/H_i$, the group $H_{i+1} \cap H^0/H_i \cap H^0$ can be considered as a normal subgroup of H_{i+1}/H_i: it is again abelian or simple. The series $1 = H_0 \lhd H_1 \cap H^0 \lhd \cdots \lhd H_{n-1} \cap H^0 \lhd H^0 \lhd G$ has the required properties, but maybe the last factor G/H^0, which is bounded.

We now show inductively on the length n of the chain that the previous series can be refined into a new series where every factor is abelian, simple or finite. Because a finite group has a series whose factor are either simple or abelian, this is sufficient. If n equals 1, then H^0 is trivial so H is finite by the Compactness Theorem. For the induction step, we note that $H^0/H^0 \cap H_{n-1}$ is isomorphic to $H^0.H_{n-1}/H_{n-1}$, which is either abelian or simple.

First case: $H^0.H_{n-1}/H_{n-1}$ is simple. Then $H^0.H_{n-1}$ is definable by Proposition 5.2. By the Compactness Theorem, there is a definable normal subgroup K of H of finite index such that $H^0.H_{n-1}$ equals $K.H_{n-1}$. It follows that $K/K \cap H_{n-1}$ is isomorphic to $H^0/H^0 \cap H_{n-1}$ hence simple. Note that $K \cap G/K \cap H_{n-1}$ has bounded index in $K/K \cap H_{n-1}$ hence is also simple. As $K \cap H_{n-1}/H^0 \cap H_{n-1}$ is bounded, the series

$$1 = H_0 \lhd H_1 \cap H^0 \lhd \cdots \lhd H_{n-1} \cap H^0 \lhd H_{n-1} \cap K \lhd K \cap G \lhd G$$

can be refined by a series with abelian, simple or finite factors by the induction hypothesis.

Second case: $H^0/H^0 \cap H_{n-1}$ is abelian. Then, by the Compactness Theorem, there is a definable subgroup K of finite index in H, such that $K/H^0 \cap H_{n-1}$ is abelian. We may replace K by a finite intersection of H-conjugates of it and assume that K is normalised by H, hence by G. We consider the series

$$1 = H_0 \lhd H_1 \cap H^0 \lhd \cdots \lhd H_{n-1} \cap H^0 \lhd K \cap G \lhd G$$

where only the two last factor groups $K \cap G/H_{n-1} \cap H^0$ and $G/G \cap K$ have been modified. As a subgroup of $K/H^0 \cap H_{n-1}$, the group $K \cap G/H_{n-1} \cap H^0$ is abelian whereas $G/G \cap K$ is finite. This completes the proof by induction.

We have just build a series of type-definable groups $1 = G_0 \lhd G_0 \lhd \cdots \lhd G_n = G$ such that every factor is abelian or simple. By the same type of arguments, one can show inductively that each G_i may be replaced by one which is relatively definable in G_{i+1}. □

Corollary 5.4. *In a type-definable supersimple group, there is a natural number n bounding the derived length of any subnormal soluble subgroup.*

Corollary 5.5. *A locally soluble subnormal subgroup of a type-definable supersimple group is soluble.*

Lemma 5.6. *Let G be any type-definable group. Assume that G is the conjunction of definable groups H_i where i ranges over some set I. Let $R(G)$ be the soluble radical of G. If $R(G)$ is soluble, then there is a finite conjunction H_J of groups H_i such that*

$$R(G) = R(H_J) \cap G.$$

In particular, $R(G)$ is relatively definable in G.

Proof. For any finite subset J of I, we write H_J for the intersection of H_j when j ranges over J. For any finite subset J of I, if x belongs to $R(H_J) \cap G$, then x^{H_J} generates a soluble subgroup so x^G also generates a soluble subgroup and x is an element of $R(G)$. Reciprocally, if x belongs to $R(G)$, then x^G generates a soluble subgroup of derived length n say. Because $R(G)$ is soluble, the number n does not depend on x. It follows that $(x^G \cup x^{-G})^{(n)}$ equals $\{1\}$ where for a subset X of G, the set $X^{(n)}$ is defined inductively on $n \geq 0$ by

$$X^{(0)} = X, \quad X^{(1)} = [X, X] \quad \text{and} \quad X^{(n+1)} = \big[X^{(n)}, X^{(n)}\big].$$

This means that the partial type $G \times \cdots \times G$ in 2^n variables $(x_1, \ldots, x_{2^n})$ implies the formula $f(x, x_1, \ldots, x_{2^n}) = 1$ where $f(x, x_1, \ldots, x_{2^n})$ is the word defined inductively on $n \geq 1$ by

$$f(x, x_1, x_2) = [x^{x_1}, x^{x_2}] \quad \text{and}$$

$$f(x, x_1, x_2, \ldots, x_{2^{n+1}}) = [f(x, x_1, x_2, \ldots, x_{2^n}), f(x, x_{2^n+1}, x_{2^n+2}, \ldots, x_{2^{n+1}})]$$

By the Compactness Theorem, there is a finite subset J of I such that the same formula

$$f(x, x_1, \ldots, x_{2^n}) = 1$$

is implied by the formula

$$(x_1, \ldots, x_{2^n}) \in H_J \times \cdots \times H_J$$

This means precisely that x^{H_J} generates a soluble group of derived length at most n, hence x is an element of $R(H_J) \cap G$. □

Corollary 5.7. *The soluble radical of a group G which is type-definable in a supersimple theory is soluble and relatively definable in G.*

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